International Journal of Theoretical Physics, Vol. 10, No. 3 (1974), pp. 197–216

Internal Geometrical Structures on Minkowski Space-Time and Yang–Mills Symmetries†

M. CAMENZIND

Institut für theoretische Physik, Universität Bern, Sidlerstrasse 5, 3012 *Bern*

Received: 1 *October* 1973

Abstract

Yang–Mills potentials and fields on Minkowski space-time are equivalent to connections and curvatures for corresponding principle fibre bundles over Minkowski space-time. Therefore, the usual Yang–Mills field equations, together with Bianchi's identities, define transport equations for the connection and curvature with respect to a given observer field (inertial system). This equivalence is worked out; furthermore, the notion of symmetric, particularly spherically symmetric Yang–Mills connections is transposed to the bundle description. We show that the restrictions on the Yang–Mills connection imposed by spherical symmetry are by no means as severe as assumed by Ikeda and Miyachi.

1. *Introduction*

When looking at geometrical objects defined on Minkowski space-time, we would immediately enumerate two types of quantities, tensors and quantum fields transforming in a unique way under some representation of the Lorentz group. Physical arguments and physical principles will rule out some of these quantities; e.g., there are no geometrical reasons for excluding infinite spin systems on Minkowski space-time, but physical arguments based on the mass spectrum, locality and other assumptions in a local quantum field theory will help to eliminate most of them (O'Raifeartaigh, 1970).

The above-mentioned quantities are intimately related to the Lorentz group; however, on Minkowski space-time there exists a great deal of other geometrical structures which can be always attached to a manifold, some of them were thought to be describing internal symmetries of particles (Yang & Mills, 1954; Utiyama, 1956), properties of particles independent of space-time in the sense that they are inherent, at each point, to space-time. Before asking the question

† This is based on a Thesis accepted by the Faculty of Philosophy and Exact Sciences, University of Berne.

Copyright © 1974 Plenum Publishing Company Limited. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of Plenum Publishing Company Limited.

which internal symmetry group is the adequate one in fitting the stand of to-day's experimental data, I would like, therefore, to turn the question to which internal properties can be constructed on Minkowski space-time and, then, which can be ruled out on the strength of physical and symmetry principles. The first such internal geometrical structures were welded upon Minkowski space-time by Yang & Mills (1954) in attempting to tie together the isospin behaviour of particles with internal properties of Minkowski space-time; the internal structure of the Yang-Mills theory was shown to be equivalent to a principal fibre bundle structure over Minkowski space-time (Kerbrat, 1970; Loos, 1967).

In the following, we shall work out the geometrical contents of such a general 'curvature theory' over Minkowski space, and especially the meaning of the principle of minimal interaction which is commonly used as the fundamental construction principle for these types of 'fields'. In order to get more insight into the meaning of this principle, let us first give the interpretation of this principle in the gravitational case. In Einstein's theory, gravitation is described by the properties of Lorentzian manifolds, characterised, say, by the corresponding metric or the Christoffel symbols. Assume we were to translate an interaction theory (a non-gravitational one), expressed in terms of differential equations involving tensor fields and quantum fields on Minkowski space-time, into the corresponding theory on a curved space-time, then the principle of minimal interaction contains the information of how to construct the interaction theory in this case: replace ordinary derivatives of the tensor or quantum fields by covariant ones. But it never claims how to express the gravitation theory itself.

A general Yang-Mills theory proceeds to replace a Lorentz covariant object φ (transforming under some representation of the Lorentz group) by a set of such objects φ^a, $a = 1, 2 \ldots, N$, which transform at the same time also under a gauge group G

$$\varphi^{a'}(x) = S^{-1a}{}_b \varphi^b(x), \qquad S = S(x) \in G \tag{1.1}$$

φ itself obeys a differential equation, most of them are Dirac-like equations,

$$L(\varphi) = 0$$

Then the equation, including fields with internal degrees of freedom, follows immediately in the spirit of the minimal interaction principle by

$$\partial_\mu \to \partial_\mu \delta^a{}_b + \Gamma_\mu{}^a{}_b \tag{1.2}$$

where the $\Gamma_\mu{}^a{}_b$ represent this new internal structure. Let me add some remarks to the above description:

First of all, (1.1) only makes sense if we define the geometrical meaning of the φ^a's: according to the transformation property (1.1) under gauge transformations the φ^a are the components of a 'vector', called internal vector, with respect to some internal basis attached at each point on Minkowski space-time. Therefore, the geometrical background for (1.1) is the following: On Minkowski space-time, there is defined a principal fibre bundle $P(V, G, \pi)$ determined by the structure group G, projection π and spanned by the set of all bases $\{\psi_a\}$ (Kobayashi & Nomizu, 1963; Loos, 1967); the principle of minimal interaction (1.2) means that the original Dirac theory without internal degrees of freedom goes over to a theory with coupling to the internal space P. This coupling is given in terms of the connection coefficients $\Gamma_\mu{}^a{}_b$ with respect to a special basis, cross-section, $\{x, \psi_a\}$, and a given connection in the principal fibre bundle P, usually written as covariant derivatives

$$\nabla_\mu \psi_a = \Gamma_\mu{}^b{}_a \psi_b, \qquad \forall \mu, a \tag{1.3}$$

The principle of minimal interaction never tells us how to construct this internal connection. In our geometrical language, we have more information about the connection whenever we know how a special internal frame—the elements of P in the following are called Yang-Mills frames or only frames, whenever there is no confusion with the elements of the linear frame bundle over Minkowski space—say ψ, is transported along four different directions in space-time with respect to horizontal frames in these directions. Especially if there exists a frame $\bar{\psi}$ which is parallely transported in four independent directions, i.e.

$$\nabla_X \bar{\psi}_a = 0, \qquad \forall X \in TV, \qquad \forall a, \tag{1.4}$$

then the connection will be flat, i.e. the curvature of the connection vanishes.

These Yang-Mills geometries defined on Minkowski space-time can be treated more easily than the corresponding geometries on curved space-times because of the flatness of the underlying base manifold, i.e. because the description of the observers, inertial frames, assumes a partially simple form; moreover, interactions between these internal geometries and gravitation are excluded from the given picture.

In Section 2 we summarise the differential geometry of the Yang-Mills bundles; the equivalence between the physical formulation of a Yang-Mills theory and the connection approach for the corresponding principal fibre bundle is essentially based on the transformation law of the connection form under transformations between local cross-sections. Section 3 deals with the dynamics for a Yang-Mills connection and the corresponding transport of curvature and associated elements along an observer trajectory on Minkowski space-time. This dynamics will be called curvature dynamics of the internal geometries. The dynamics of these structures turns out to be self-determining in the sense that only the transport of the time-like part of the curvature, $\tilde{R}_{0i}$, is not specified by the structure of the connection itself.

A rough classification scheme for the Yang-Mills geometries is usually based on the holonomy structure of the connection for a particular principal fibre bundle (see Section 4); symmetries for a connection were expected to reduce the infinitesimal holonomy group (Ikeda & Miyachi, 1962; Loos, 1965). Therefore, in order to analyse these restrictions, we shall work out an exact definition of a *Yang-Mills symmetry* in Section 5 over the notion of bundle automorphisms. In general, such symmetry transformations imply restrictions on the constituents of the Lie algebra of the infinitesimal holonomy group.

Particularly, symmetry transformations induced by a subgroup of the Poincaré group (considered as the isometry group of Minkowski space-time) are investigated, and this will enable us to show that the infinitesimal holonomy group for a spherically symmetric Yang-Mills connection will not necessarily be Abelian. A specific example will be published in a forthcoming paper.

2. *Differential Geometry of Yang-Mills Bundles over Minkowski Space-Time*

In the spirit of differential geometry of fibre bundles (Kobayashi & Nomizu, 1963; Trautman, 1970) vector fields can be considered as functions over the corresponding principal fibre bundle P, transforming in a special way under fibre transformations. Therefore, the basic object for the internal structures is P itself, and not an associated structure. Since the notion of a connection in a principal fibre bundle is the most fundamental for the following discussions, let us summarise the most important aspects. A connection Γ of P is a distribution of a subspace $H_u \subset T_u P$ to every $u \in P$ so that π_* defines a vector space isomorphism $\pi_* : H_u \to T_{\pi(u)} V$ and the elements of H are invariant under the right action of the structure group G on the fibres; H_u is called horizontal subspace. Since G acts transitively and effectively on the fibres of P, there exists an isomorphism between the tangent space on the fibres and the Lie algebra of the structure group G, identified in the following with $\mathbf{G}_e = T_e G$. The inverse mapping of this isomorphism defines the connection form ω with respect to a given connection Γ (as a generalisation of the canonical 1-forms on Lie groups) and, conversely, a given Lie algebra-valued 1-form ω defines in a unique way a connection Γ in P. With respect to a given cross-section σ: $U \subset V \to P$ ω is usually decomposed

$$\omega_U = (\tilde{\Gamma}_\mu{}^A E_A) dx^\mu, \qquad E_A \text{ some basis of } \mathbf{G}_e, \qquad A = 1, \ldots, \dim G \tag{2.1}$$

Every curve in the Minkowski space-time V, say x_t, $t \in [a, b]$, gives rise to a set of curves $u(t)$ in P, namely to all those which project down to x_t

$$\pi(u(t)) = x_t, \qquad t \in [a, b]$$

Now, the lifting lemma for connections on principle fibre bundles distinguishes one special type of curves in P, the horizontal curves $u(t)$. A curve $u(t)$ in P is called horizontal if $\pi(u(t)) = x_t$ and the tangent vector $\dot{u}(t)$ is horizontal for all parameter values t with respect to the given connection or, in other words,

$$\omega(\dot{u}(t)) = 0, \qquad t \in [a, b]$$

Lifting Lemma. Let a connection Γ in P be given, ω its connection form. Consider a curve x_t, $t \in [a, b]$, in V and a frame u_a for $t = a$; then there exists a unique horizontal curve $u(t)$ in P with the initial conditions $u(a) = u_a$.

The proof of this lemma is based on the transformation property for the connection form under transformation between two different curves $u(t)$ and $v(t)$ in P with

$$u(t) = v(t)g(t), \qquad g(t) \in G \text{ for every } t$$

This means, to two curves in P there is associated a unique curve in the structure group G. Then the corresponding connection forms are related by

$$\omega(\dot{u}(t)) = adg^{-1}(t)\omega(\dot{v}(t)) + g^{-1}(t)\dot{g}(t) \tag{2.2}$$

$\dot{g}(t)$ being the tangent vector to the curve $g(t)$ in G at the point $g(t)$; therefore $g^{-1}(t)\dot{g}(t) = \theta_{g(t)}(\dot{g}(t)) \in \mathbf{G}_e$ (θ is the canonical 1-form for the Lie group G). $u(t)$ is horizontal iff

$$\theta_{g(t)}(\dot{g}(t)) = A(t), \qquad A(t) \text{ a continuous curve in } \mathbf{G}_e \tag{2.3}$$

This defines a differential equation on the Lie group G with initial condition $g(a) = e$, the unity element in G; the differential equation (2.3) has always a uniquely defined solution (Kobayashi & Nomizu, 1963), in as much as we have in (2.3) a generalization of the differential equation for A = const., corresponding to the 1-parameter subgroups in G.

As a consequence of (2.2) we obtain the transformation of the connection form under cross-section transformations; say ω is locally given with respect to the cross-section $\sigma : x \to (x, \psi_a)$, $a = 1, \ldots, N$, by (2.1), and let us be given another cross-section $\sigma' : x \to (x, \psi'_a)$, then there exists a $g \in G$ for each $x \in U$

$$\psi'_a = \psi_b g^b{}_a, \qquad g = g(x) \text{ on } U$$

such that

$$\omega_{\psi'}{}^a{}_b = g^{-1a}{}_c(dg^c{}_b + \tilde{\Gamma}_\mu{}^c{}_d g^d{}_b dx^\mu) \tag{2.4}$$

In (2.4) we find the usual transformation property of the Yang–Mills potentials $\tilde{\Gamma}_\mu{}^a{}_b$ under gauge transformations (1.1). A second consequence of (2.2) is an interpretation of the covariant derivative for objects associated with the principal fibre bundle P.

Definition. α is called pseudo-tensorial p-form of type adG, G being always the structure group of P, if

(1) α is a $\mathbf{G}_e$-valued p-form on P;
(2) α is of type adG, i.e.

$$R_g^* \alpha = adg^{-1}\alpha \quad \text{for all } g \in G$$

α is a tensorial form of type adG if

$$\alpha(\tilde{X}_1, \ldots, \tilde{X}_p) = 0$$

whenever at least one of the $\tilde{X}_i \in T_u P$ is vertical.

Tensorial forms correspond to Lie algebra-valued functions $\tilde{f}$ on P, which are of type adG (Kobayashi & Nomizu, 1963). In analogy to the definition of the covariant derivative of functions on the linear frame bundle we define

Definition. If X is a vector field on V, $\tilde{f}$ some set of functions on P

$$\widetilde{\nabla_X \varphi} := \tilde{X}_u . \tilde{f}, \qquad x = \pi(u) \tag{2.5}$$

where $\tilde{X}_u$ is the horizontal lift of X to P, φ is the geometrical object on V which corresponds to $\tilde{f}$.

As an example, an internal vector field φ on an open set $U \subset V$ is determined by a set of functions $\tilde{\varphi}^a$, $a = 1, \ldots, N$, which transform under the right action R of the structure group G on the fibres $\pi^{-1}(x)$

$$\tilde{\varphi}^a(ug) = g^{-1a}{}_b \tilde{\varphi}^b(u), \qquad u \in \pi^{-1}(x) \text{ and } g \in G \tag{2.6}$$

Conversely, the set of functions $\tilde{\varphi}^a$ defines in a unique way a vector field φ with respect to a chosen cross-section $\sigma : U \to P$

$$\varphi_u := \sigma^* \tilde{\varphi} \tag{2.7}$$

Let τ^* be the parallel displacement of the fibre $\pi^{-1}(x_0)$ along the curve x_t in V; then $\tau^* : \pi^{-1}(x_0) \to \pi^{-1}(x_1)$ is an isomorphism according to the lifting lemma and the fact that τ^* commutes with the right action R on the fibres. With respect to the horizontal curve $u(t)$, definition (2.5) of the covariant derivative is equivalent to

$$\tilde{X}_u . \tilde{f} = \frac{d}{dt} \tilde{f}(u(t)) \bigg|_{t=0}$$

which means the covariant differentiation of some geometrical object is equivalent to differentiate the corresponding set of functions on P with respect to a horizontal curve in the given direction. And therefore, the 'generalised Christoffel symbols' $\Gamma_\mu{}^a{}_b$ in (1.3) and (2.1) contain the information of how a Yang–Mills frame is transported with respect to the horizontal frame in the given space-time direction X. The Yang–Mills connection Γ is completely determined by the information of how a Yang–Mills frame is transported with respect to the corresponding horizontal frames *in four independent space-time directions.* In general, we have only this information for one time-like direction representing the trajectory of an observer. As we shall see in the next section, the dynamics for such a curvature theory will completely determine the space-like parts of the connection together with a given current.

If α is a pseudo-tensorial form of type adG, we define the horizontal part of α by

$$(\text{hor }\alpha)(\tilde{X}_1, \ldots, \tilde{X}_p) := \alpha(\text{hor }\tilde{X}_1, \ldots, \text{hor }\tilde{X}_p) \tag{2.8}$$

especially, $\Omega := \text{hor } d\omega$ defines the curvature 2-form and is a tensorial 2-form of type adG which enables the decomposition

$$\Omega = \frac{1}{2} \tilde{R}_{\rho\sigma} dx^\rho \wedge dx^\sigma \tag{2.9}$$

with respect to a local cross-section σ. $\tilde{R}_{\rho\sigma}$ is now a skew-symmetric $\mathbf{G}_e$-valued tensor, related to the connection form by the second structure equation on a principal fibre bundle

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega \tag{2.10}$$

or

$$\tilde{R}_{\rho\sigma} = \partial_\rho \Gamma_\sigma - \partial_\sigma \tilde{\Gamma}_\rho + [\tilde{\Gamma}_\rho, \tilde{\Gamma}_\sigma] \tag{2.11}$$

(,) being the Lie bracket in the Lie algebra $\mathbf{G}_e$ and at the same time an expression for the exterior product of 1-forms. The tensorial form α can be decomposed locally as

$$\alpha = \frac{1}{p!} Q_{\rho_1 \ldots \rho_p} dx^{\rho_1} \ldots dx^{\rho_p}$$

and correspondingly its covariant derivative

$$\tilde{\nabla}_\lambda \tilde{Q}_{\rho_1 \ldots \rho_p} = \partial_\lambda \tilde{Q}_{\rho_1 \ldots \rho_p} + [\tilde{\Gamma}_\lambda, \tilde{Q}_{\rho_1 \ldots \rho_p}] \tag{2.12}$$

which satisfies the following two identities

$$\tilde{\nabla}_{[\lambda} \tilde{\nabla}_{\mu]} Q = \frac{1}{2}[\tilde{R}_{\lambda\mu}, \tilde{Q}] \qquad \text{for a 0-form } \alpha \tag{2.13}$$

$$\tilde{\nabla}_{[\lambda} \tilde{\nabla}_{\mu]} \tilde{Q}^{\lambda\mu} = \frac{1}{2}[\tilde{R}_{\lambda\mu}, \tilde{Q}^{\lambda\mu}] \qquad \text{for a 2-form } \alpha \tag{2.14}$$

Bianchi identities read as

$$d\Omega = \frac{1}{2}[\Omega, \omega] + \frac{1}{2}[\omega, \Omega] \tag{2.15}$$

or locally expressed

$$\tilde{\nabla}_{[\mu} \tilde{R}_{\rho\sigma]} = 0 \tag{2.15a}$$

3. *The Dynamics for the Yang–Mills Connection*

In this section we shall relate the dynamics for the connection of a Yang–Mills bundle $P(V, G, \pi)$ over Minkowski space-time in an intrinsic manner to the observer systems. Take a time-like future pointing vector field V^μ on V, $V^\mu \partial_\mu = d/dt$; then a Yang–Mills frame $\bar{u} = (x, \psi_a)$ is horizontal along V^μ for a given connection Γ in P iff

$$\tilde{\Gamma}_\mu V^\mu = 0, \quad \tilde{\Gamma}_\mu \quad \text{are the connection coefficients with respect to the cross-section } \bar{u} \tag{3.1}$$

The contraction of the structure equation (2.11) with V^μ and a second vector field K^μ orthogonal to V^μ gives

$$\frac{d}{dt}(\tilde{\Gamma}_\alpha K^\alpha) = \tilde{R}_{\alpha\beta} V^\alpha K^\beta \tag{3.2}$$

or especially for an adjusted coordinate system with $V^\mu = \delta_0{}^\mu, K^\alpha = \delta_i{}^\alpha$

$$\frac{d}{dt}\tilde{\Gamma}_i = \tilde{R}_{0i} \qquad \text{and} \qquad \tilde{\Gamma}_0 = 0 \tag{3.2a}$$

meaning that $\tilde{R}_{0i}$ determine the dynamics for the connection coefficients $\tilde{\Gamma}_i$, which we call in the following the space-like part of the connection form ω, $\tilde{R}_{0i}$ correspondingly the time-like part of the curvature form.

A second dynamical equation follows immediately from the Bianchi identities (2.15) by contraction with $V^\mu, K^\alpha, L^\lambda$ being orthogonal to V^μ and K^α:†

$$\partial_\mu \tilde{R}_{\alpha\lambda} + \partial_\lambda \tilde{R}_{\mu\alpha} + \partial_\alpha \tilde{R}_{\lambda\mu} + [\tilde{\Gamma}_\mu, \tilde{R}_{\alpha\lambda}] + [\tilde{\Gamma}_\lambda, \tilde{R}_{\mu\alpha}] + [\tilde{\Gamma}_\alpha, \tilde{R}_{\lambda\mu}] = 0$$

$$\begin{aligned} V^\mu \partial_\mu \tilde{R}_{\alpha\lambda} K^\alpha L^\lambda = {} & -L^\lambda \partial_\lambda \tilde{R}_{\mu\alpha} V^\mu K^\alpha - K^\alpha \partial_\alpha \tilde{R}_{\lambda\mu} V^\mu L^\lambda \\ & - [\tilde{\Gamma}_\lambda L^\lambda, \tilde{R}_{\mu\alpha} V^\mu K^\alpha] - [K^\alpha \tilde{\Gamma}_\alpha, \tilde{R}_{\lambda\mu} V^\mu L^\lambda] \end{aligned} \tag{3.3}$$

$$\frac{d}{dt}\tilde{R}_{\alpha\lambda} K^\alpha L^\lambda = F(\tilde{\Gamma}, \tilde{R}, \partial\tilde{R}) \tag{3.4}$$

Especially, for $V^\mu = \delta_0{}^\mu, K^\alpha = \delta_k{}^\alpha, L^\lambda = \delta_i{}^\lambda$

$$\frac{d}{dt}\tilde{R}_{ki} = F_{ki}(\tilde{\Gamma}, \tilde{R}_{0k}) = -\tilde{\nabla}_i \tilde{R}_{0k} + \tilde{\nabla}_k \tilde{R}_{0i} \tag{3.4a}$$

Therefore, according to (3.2a) and (3.4a) the dynamics for the space-like part of the connection and for the space-like part of the curvature is completely determined by the time-like part of the curvature, $\tilde{R}_{0i}$, considered as a field and determined with respect to a field of observers; these components are often called 'electrical' components of the curvature; in other words, the only 'free observable' for such a geometrical curvature theory is the time-like part of the curvature, since both, the space-like connection and the space-like curvature, are predetermined by the assumed structure of a principal fibre bundle. In order to close up the dynamical system for the internal geometry, we expect the existence of a current-like object $\tilde{J}_i$, $\mathbf{G}_e$-valued and of type *adG*, coupled to the time-like curvature

$$\frac{d}{dt}\tilde{R}_{0i} \sim \tilde{J}_i, \qquad \text{or} \qquad V^\mu \partial_\mu \tilde{R}_{\mu\alpha} V^\mu K^\alpha \sim \tilde{J}_\alpha K^\alpha \tag{3.5}$$

all quantities expressed with respect to a horizontal cross-section σ in the direction of V^μ. In general, this current $\tilde{J}_i$, governing over the time variation of $\tilde{R}_{0i}$, will have an internal part related to the internal geometry of the principal fibre bundle, which a comparison with (3.4) will suggest.

† *V, K* and *L* are always given in terms of inertial coordinates.

Working all the time with a horizontal cross-section σ for the Yang–Mills bundle P would emphasise in an inadequate way the role of these special frames and, moreover, a horizontal frame for the direction V^μ will be in general no longer horizontal for another time-like direction V'^μ. Therefore, the dynamics (3.2), (3.4) and (3.5) should not depend on the chosen frame or, in other words, on the chosen observer. This implies that the differential equations should be 'internal covariant' and the current $\tilde{J}_i$ of type adG. But we see we could interpret this principle of 'internal covariance' in the spirit of the principle of covariance in General Relativity, since the deeper meaning of even the principle of 'internal covariance' is that the whole dynamics for our internal curvature theory should not depend on the special observer system even on the background of Minkowski space-time. We choose in the following a cross-section which is not horizontal in the direction V^μ. The second structure equation is a covariant equation

$$V^\mu \partial_\mu(\tilde{\Gamma}_\alpha K^\alpha) + [\tilde{\Gamma}_\mu V^\mu, \tilde{\Gamma}_\alpha K^\alpha] = \tilde{R}_{\mu\alpha} V^\mu K^\alpha + K^\alpha \partial_\alpha(\tilde{\Gamma}_\mu V^\mu)$$

$$\frac{d}{dt}(\tilde{\Gamma}_\alpha K^\alpha) = \tilde{R}_{\mu\alpha} V^\mu K^\alpha + K^\alpha \partial_\alpha(\tilde{\Gamma}_\mu V^\mu) - [\tilde{\Gamma}_\mu V^\mu, \tilde{\Gamma}_\alpha K^\alpha] \tag{3.6}$$

or in terms of our special directions

$$\frac{d}{dt}\tilde{\Gamma}_i = \tilde{R}_{0i} + \partial_i \tilde{\Gamma}_0 - [\tilde{\Gamma}_0, \tilde{\Gamma}_i] \tag{3.6a}$$

The dynamics for the space-like connection is contained in $\tilde{R}_{0i}$ and $\tilde{\Gamma}_0$, the latter gives the transport of the chosen frame along the direction V^μ. Bianchi identities are also internal covariant equations

$$V^\mu \tilde{\nabla}_\mu(\tilde{R}_{\alpha\lambda} K^\alpha L^\lambda) = -L^\lambda \tilde{\nabla}_\lambda(\tilde{R}_{\mu\alpha} K^\alpha V^\mu) + K^\alpha \tilde{\nabla}_\alpha(\tilde{R}_{\mu\lambda} V^\mu L^\lambda) \tag{3.7}$$

or especially

$$\frac{d}{dt}\tilde{R}_{ki} = -\tilde{\nabla}_i \tilde{R}_{0k} + \tilde{\nabla}_k \tilde{R}_{0i} - [\Gamma_0, \tilde{R}_{ki}] \tag{3.7a}$$

The remaining problem is to find the internal part of the current in (3.5) such that the possibilities for the time-like curvature are not too severely restricted. The idea to close the geometrical theory with a covariant 'conservation equation' for the external part of the current $\tilde{J}^{\text{ex}}_\mu$ of the type

$$\tilde{\nabla}_\mu \tilde{J}^{\text{ex}\,\mu} = 0 \tag{3.8}$$

suggests the use of the Yang–Mills type equations

$$\partial_\mu \tilde{R}^{\mu\alpha} + [\tilde{\Gamma}_\mu, \tilde{R}^{\mu\alpha}] = g\tilde{J}^\alpha \tag{3.9}$$

which turn out to be a generalisation of Maxwell's equations for principal fibre bundles with non-Abelian structure groups (Uzes, 1971) (g is some coupling constant). The 'conservation equation' (3.8) proceeds in this case automatically from the identity (2.14). Furthermore, (3.9) is a covariant equation and corresponds to the 'dual Bianchi equations'

$$\tilde{\nabla}[_\mu {}^*\mathrm{R}_{\rho\sigma}] = g^*\tilde{J}_{\mu\rho\sigma} \tag{3.10}$$

$$^*\tilde{\mathrm{R}}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\rho\sigma}\tilde{R}^{\rho\sigma} \tag{3.11}$$

This would imply for the time-like curvature $\tilde{R}_{0i}$, which corresponds to $^*\tilde{R}_{k\ell}$,

$$\partial_\mu {}^*\tilde{R}_{\rho\sigma} + \partial_\rho {}^*\tilde{R}_{\sigma\mu} + \partial_\sigma {}^*\tilde{R}_{\mu\rho} + [\tilde{\Gamma}_\mu, {}^*\tilde{R}_{\rho\sigma}] + [\tilde{\Gamma}_\rho, {}^*\tilde{R}_{\sigma\mu}] + [\tilde{\Gamma}_\sigma, {}^*\tilde{R}_{\mu\rho}] = \frac{g}{2}\epsilon_{\mu\rho\sigma\alpha}\tilde{J}^\alpha \tag{3.12}$$

then

$$V^\mu\tilde{\nabla}_\mu({}^*\tilde{R}_{\alpha\lambda}K^\alpha L^\lambda) = -L^\lambda\tilde{\nabla}_\lambda({}^*\tilde{R}_{\mu\alpha}V^\mu K^\alpha) + K^\alpha\tilde{\nabla}_\alpha({}^*\tilde{R}_{\mu\lambda}V^\mu L^\lambda) - g\tilde{J}_{\mu\alpha\lambda}V^\mu K^\alpha L^\lambda \tag{3.13}$$

or especially

$$\begin{aligned}\tilde{\nabla}_0 {}^*\tilde{R}_{ki} &= -\tilde{\nabla}_i {}^*\tilde{R}_{0k} + \tilde{\nabla}_k {}^*\tilde{R}_{0i} - g^*\tilde{J}^{\mathrm{ex}}{}_{0ki} \\ &= -F'_{ki} - g^*\tilde{J}^{\mathrm{ex}}_{0ki}\end{aligned} \tag{3.13a}$$

The vacuum Yang-Mills equations (3.9) follow from a variational principle for the Yang-Mills connection (Kerbrat, 1970)

$$I = \mathrm{Tr}\int_D \tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}d^4x \tag{3.14}$$

with the associated energy-momentum tensor for the connection

$$T^{\mu\nu} = \mathrm{Tr}\left\{\tilde{R}^{\mu\rho}\tilde{R}_\rho{}^\nu + \frac{1}{4}g^{\mu\nu}\tilde{R}_{\rho\sigma}\tilde{R}^{\rho\sigma}\right\} \tag{3.15}$$

Remark. Equations (3.7a) and (3.13a) show that the evolution of the components of the internal curvature along an observer trajectory is governed by the variation of the dual curvature elements in space-like directions; from a field theoretical point of view, these elements are determined by the field equations; for our observer oriented point of view, these elements satisfy further differential equations which follow from the elements of the Lie algebra of the infinitesimal holomy group (cf. next section). With respect to a horizontal Yang-Mills frame we find, e.g.,

$$\begin{aligned}\frac{d}{dt}\tilde{\nabla}_i\tilde{R}_{0k} &= \frac{d}{dt}\tilde{f}_{i0k} \\ &= \partial_i\left(\frac{d}{dt}\tilde{R}_{0k}\right) + \left[\frac{d}{dt}\tilde{\Gamma}_i, \tilde{R}_{0k}\right] + \left[\tilde{\Gamma}_i, \frac{d}{dt}\tilde{R}_{0k}\right] \\ &= \tilde{\nabla}_i\left(\frac{d}{dt}\tilde{R}_{0k}\right) + \left[\tilde{R}_{0i}, \tilde{R}_{0k}\right]\end{aligned} \tag{3.16}$$

which follows also immediately from the identity (cf. equation (4.4))

$$(\tilde{\nabla}_0\tilde{\nabla}_i - \tilde{\nabla}_i\tilde{\nabla}_0)\tilde{R}_{0k} = [\tilde{R}_{0i}, \tilde{R}_{0k}]$$

If we knew a solution for the dynamical equations of the curvature with corresponding initial conditions, then we would have the right-hand side of (3.16) except the first term; therefore we need evolution equations for higher order elements of the holonomy algebra. As a consequence, we get a countable set of differential equations associated with a given observer trajectory in space-time for all the elements spanning the subspaces m_k (defined in the next section) of the Lie algebra of the holonomy group (for all $k = 0, 1, \ldots$).

4. *Holonomy Structure of Yang-Mills Bundles*

Since the curvature structure of Yang–Mills bundles is a less intuitive structure than the geometry of the linear or the Lorentz frame bundle over space-time, we have to find a rough classification scheme for these geometries. The most obvious one is based on the infinitesimal holonomy group and its Lie algebra which is generated by the curvature elements and higher order covariant derivatives of the curvature (Kobayaski & Nomizu, 1963; Lichnerowicz, 1962; Ikeda & Miyachi, 1962; Loos, 1965). Symmetries for a Yang–Mills connection were expected to reduce the number of internal degrees of freedom in the sense that the infinitesimal holonomy group is, e.g., perfect in some cases, at least in the case of spherically symmetric holonomy groups for source-free regions (Ikeda & Miyachi, 1962; Loos, 1965). The lack of a first structure equation for a Yang–Mills connection has the consequence that all the usual 'symmetries' of the Riemannian curvature tensor $R^{\alpha}{}_{\beta\rho\sigma}$ have no sense and do not exist, because internal labels and space-time indices cannot be mixed up; they have a completely different meaning. A Yang–Mills curvature cannot be considered in general as a fourth-rank tensor over space-time.

In the following, we define a set of spaces $m_k(u)$, u being a fixed frame on P (Kobayaski & Nomizu, 1963):

(1) $m_0(u) = \{\tilde{\Omega}_u(\tilde{V}, \tilde{W}); \tilde{V}, \tilde{W} \in H_u\}$

(2) let $\tilde{f}$ denote a set of functions of type adG, $\mathbf{G}_e$-valued, of the form
$\tilde{f} = \tilde{X}_1 \ldots \tilde{X}_k(\tilde{\Omega}(\tilde{V}, \tilde{W})); \tilde{X}_1, \ldots, \tilde{X}_k, \tilde{V}, \tilde{W}$ horizontal,
$k = 1, 2, \ldots$

$$m_k(u) := \{\tilde{f}; \tilde{f} \in m_{k-1}(u) \text{ or } \tilde{f} \text{ of form (4.1)}\}, k = 1, 2, \ldots \tag{4.1}$$

(3) $\mathbf{H}'_e(u)$ as the union of all the spaces $m_k(u)$.

If u is a local cross-section, then the functions of type (k) correspond to elements of the form

$$X_1^{\mu_1} \ldots X_k^{\mu_k} \tilde{\nabla}_{\mu_1} \ldots \tilde{\nabla}_{\mu_k} \tilde{R}_{\rho\sigma} V^{\rho} W^{\sigma} \tag{4.2}$$

In general, for C^{∞} – connections, the set $\mathbf{H}'_e(u)$ is a sub-algebra of the Lie algebra of the local holonomy group (Kobayashi & Nomizu, 1963), and the two algebras are identical in the case of analytic connections (Lichnerowicz, 1962). Especially, we find a relation

Lemma 1.

$$[m_k(u), m_s(u)] \subseteq m_{k+s+2}(u), \qquad \forall k, s \tag{4.3}$$

which follows, e.g., from the local identity for tensorial 2-forms

$$\tilde{\nabla}_{[\lambda}\tilde{\nabla}_{\mu]}\tilde{\alpha}_{\rho\sigma} = \frac{1}{2}[\tilde{R}_{\lambda\mu}, \tilde{\alpha}_{\rho\sigma}] \tag{4.4}$$

For the next section, we prepare a fundamental lemma for functions of type (4.1), locally represented by (4.2) (Kobayashi & Nomizu, 1963).

Lemma 2. (1) If $\tilde{X}$ is a vector field on P, invariant under the right action R (e.g. induced by a local symmetry transformation), then $\tilde{X}\,.\,\tilde{f}$ is also a function of type adG.

(2) For any vector field $\tilde{X}$:

$$\text{ver}\,\tilde{X}\,.\,\tilde{f} = -[\omega_u(\tilde{X}), \tilde{f}(u)] \tag{4.5}$$

(3) For any horizontal vector field $\tilde{X}$ and $\tilde{Y}$ on P:

$$\text{ver}\,[\tilde{X}, \tilde{Y}]_u\,.\,\tilde{f} = 2[\tilde{\Omega}_u(\tilde{X}, \tilde{Y}), \tilde{f}(u)] \tag{4.6}$$

(1) and (2) are immediate consequences of the definition of a connection.
(3) Uses the second structure equation

$$\tilde{\Omega}_u(\tilde{X}, \tilde{Y}) = d\omega_u(\tilde{X}, \tilde{Y}), \text{ if } \tilde{X} \text{ and } \tilde{Y} \text{ are horizontal,}$$

$$= -\frac{1}{2}\omega_u([\tilde{X}, \tilde{Y}])$$

Then (4.5) with $\tilde{X} \to [\tilde{X}, \tilde{Y}]$ gives (4.6).

5. *Yang-Mills Symmetries*

The idea for the definition of a bundle symmetry transformation is based on the bundle automorphisms $\tilde{f}: P(V, G, \pi) \to P(V, G, \pi)$ which consist in general of three different mappings; a bundle automorphism should map fibres into fibres, i.e. if $u, u' \in \pi^{-1}(x)$, then $\exists$ mappings f_P and f_V such that

$$f_P: \pi^{-1}(x) \to \pi^{-1}(f_V(x))$$

$f_P(u)$ and $f_P(u')$ belong to the same fibre. But because of the structure of a principal fibre bundle $\exists$ an element $\tilde{g} \in G : f_P(u') = f_P(u)\tilde{g}$, and a $g \in G : u' = R_g u$. Then $g \to \tilde{g}$ defines a homomorphism $h: G \to G$.

Definition (Kobayashi & Nomizu, 1963). A bundle automorphism $\tilde{f}: P(V, G, \pi) \to P(V, G, \pi)$ consists of

(1) a diffeomorphism $f_P: P \to P$ such that

(2) $f_P(R_g u) = f_P(u)h(g)$, $u \in P$, $g \in G$, and $h: G \to G$ is a homomorphism.

Because of (2) f_P maps fibres into fibres, and to f_P there corresponds a diffeomorphism of V, say f_V, such that the following diagram is commutative

$$\begin{array}{ccc} P & \xrightarrow{f_P} & P \\ \pi\downarrow & & \downarrow\pi \\ V & \xrightarrow{f_V} & V \end{array}$$

The automorphism $\tilde{f}$ is at the same time a mapping for the corresponding connection, it maps the horizontal subspace at u into a horizontal subspace at $f_P(u)$; for a given point $u' \in \pi^{-1}(f_V(x))$, we can find, in general, only a $u \in \pi^{-1}(x)$ and a $g \in G$ (since f_P has not to be onto) with

$$u' = f_P(u)g = R_g(f_P(u))$$

Then we define the new horizontal subspace at u', $H_{u'}$, by

$$H_{u'} = R_{g*}(f_{P*}(H_u))$$

and we obtain in general a new connection Γ' in P, independent of the choice of our u and the corresponding g.

Lemma 3 (Kobayashi & Nomizu, 1963). If ω and Ω are the connection form and the curvature form of Γ, respectively, then the corresponding connection form ω' and curvature form Ω' in Γ' are related by

$$h_* \circ \omega = f_P{}^*\omega' \qquad (5.1)$$

or

$$h_*(\omega(\tilde{X})) = \omega'(f_{P*}\tilde{X}), \qquad \tilde{X} \in T_uP, \qquad u \in P$$

$$h_* \circ \Omega = f_P{}^*\Omega' \qquad (5.2)$$

or in other words the diagram

$$\begin{array}{ccc} TP & \xrightarrow{\omega} & \mathbf{G}_e \\ f_{P*}\downarrow & & \downarrow h_* \\ TP & \xrightarrow{\omega'} & \mathbf{G}_e \end{array}$$

is commutative. The mapping h associates to a 1-parameter subgroup in G a new 1-parameter subgroup, and therefore, h_* transforms the generator of this 1-parameter subgroup into a new generator.

Definition. We call a bundle automorphism $\tilde{f}$ bundle symmetry, or P-symmetry, if $\tilde{f}$ maps Γ into Γ, Γ is said to be invariant under $\tilde{f}$; this means $\tilde{f}$ maps the horizontal subspace at u into the horizontal subspace at $f_P(u)$, or

$$H_{f_P(u)} = f_{P*}(H_u)$$

Therefore, if Γ is invariant under $\tilde{f}$, we get, as a consequence of (5.1) and (5.2),

$$\omega(\tilde{X}) = \omega(f_{P*}\tilde{X});\ \tilde{X} \in T_u P$$

$$\Omega(\tilde{X}, \tilde{Y}) = \Omega(f_{P*}\tilde{X}, f_{P*}\tilde{Y}); \qquad \tilde{X}, \tilde{Y} \in T_u P$$

In general, the diffeomorphism f_V will map the loop space $C(x)$ at $x \in V$ into the loop space $C(f_V(x))$ at the point $f_V(x)$. $\tilde{f}$ is now a P-symmetry, iff the corresponding bundle transformation f_P maps horizontal curves into horizontal curves:

$$(x_t, \Psi_{ta}) \text{ horizontal loop} \xrightarrow{\tilde{f}} (f_V(x_t), f_P(\Psi_{ta})) \text{ horizontal}$$

In the following we shall look at P-symmetries which are induced by isometries of the Minkowski space-time; since the elements of a Yang–Mills bundle are at the same time Lorentz covariant, a transformation in V, say e.g.

$$f_V : x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

induces at the same time a fibre transformation

$$f_P\!: \psi_a(x) \to U(\Lambda)\psi_a(\Lambda x), \qquad \forall a = 1, \ldots, N$$

$U(\Lambda)$ being the representation of the Lorentz group for the fields ψ_a. Consider a 1-parameter transformation on Minkowski space-time, say f_t, represented by the vector field X; to f_t there corresponds a 1-parameter transformation in P, f_{tP}, which transforms the chosen frame $\{\psi_a\}$ into

$$\psi_{ta} = f_{tP}(\psi_a), \qquad \forall_a, \qquad \pi(\psi_{ta}) = f_t(x_0), \qquad \pi(\psi_a) = x_0$$

Denote by $\{\bar{\psi}_a\}$ the corresponding horizontal frame with

$$\bar{\psi}_a = \psi_{0a}, \qquad \forall a, \qquad \text{and} \qquad \psi_{ta} = \bar{\psi}_{tb} g_a{}^b(t)$$

The the transformation law for the connection form under cross-section transformations

$$\omega(\dot{\psi}_t) = adg^{-1}(t)\, \omega(\dot{\bar{\psi}}_t) + g^{-1}(t)\dot{g}(t)$$

implies, since $\omega(\dot{\bar{\psi}}) = 0$,

$$\omega(\dot{\psi}_t) = g^{-1}(t)\dot{g}(t)$$

$\dot{\psi}_t$ is related to $\dot{\psi}_0$ by the transformation f_{tP}, and we assume furthermore Γ to be invariant under this bundle automorphism; therefore

$$\omega(\dot{\psi}_t) = \omega(f_{tP*}(\dot{\psi}_0)) = \omega(\dot{\psi}_0) = A \in \mathbf{G}_e, \qquad \forall t, \qquad A = \text{const.}$$

and A determines a 1-parameter subgroup of the structure group G.

Lemma 4. Let $\tilde{f}_t$ be a 1-parameter group of bundle automorphism induced by a space-time symmetry X, which leaves the Yang–Mills connection Γ invariant. We express the connection form locally with respect to the frame $\{x, \psi_{ta}\}$, $\omega = \tilde{\Gamma}_\mu dx^\mu$. Then $\tilde{\Gamma}_\mu X^\mu$ is constant on V and generates a 1-parameter subgroup of G, which gives the transformation between the transformed frame ψ_t and the horizontal frame $\bar{\psi}_t$ in the direction of X.

This Lemma enables us to define a linear mapping between the Lie algebra of a Minkowski transformation group I and the Lie algebra of the structure group G of a Yang-Mills bundle.

Lemma 5. Let I be a group of Minkowski transformations which induce a corresponding group of bundle automorphisms $\tilde{I}$. Suppose the connection is invariant under $\tilde{I}$; then

$$X \to \tilde{\Gamma}_\mu X^\mu, \qquad X \in \mathscr{H}(I)$$

defines a linear mapping $\bar{\omega}: \mathscr{H}(I) \to \mathbf{G}_e$, $\mathscr{H}(I)$ is the Lie algebra of I, with respect to the local cross-section $\{x, \psi_{ta}\}$ of P, or invariantly expressed

$$X \to \omega_\psi(\tilde{X}), \qquad \tilde{X} \text{ generates } \tilde{f}$$

Lemma 6. Under the condition of the above lemma we find

(1) If $\tilde{Y}$ is horizontal on P, $\tilde{X}$ induced by a P-symmetry $\tilde{f}$, then $[\tilde{X}, \tilde{Y}]$ is horizontal;

(2) the curvature satisfies

$$\Omega_\psi(\tilde{X}, \tilde{Y}) = [\omega_\psi(\tilde{X}), \omega_\psi(\tilde{Y})] - \omega_\psi([\tilde{X}, \tilde{Y}]); \qquad \tilde{X}, \tilde{Y} \in \mathscr{H}(\tilde{I})$$

Proof. (1) $\tilde{X}.(\omega(\tilde{Y})) = L_{\tilde{X}}\omega(\tilde{Y}) + \omega([\tilde{X}, \tilde{Y}])$ since ω is invariant under $\tilde{f}$, $L_{\tilde{X}}\omega = 0$, but $\tilde{Y}$ is horizontal $\omega(\tilde{Y}) = 0$, therefore $\omega((\tilde{X}, \tilde{Y})) = 0$

(2) From the second structure equation we have

$$\Omega(\tilde{X}, \tilde{Y}) = d\omega(\tilde{X}, \tilde{Y}) + \frac{1}{2}[\omega(\tilde{X}), \omega(\tilde{Y})]$$

$$= \frac{1}{2}\{\tilde{X}.\,\omega(\tilde{Y}) - \tilde{Y}.\,\omega(\tilde{X}) - \omega([\tilde{X}, \tilde{Y}]) + [\omega(\tilde{X}), \omega(\tilde{Y})]\}$$

and from the invariance condition

$$L_{\tilde{X}}\omega(\tilde{Y}) = \tilde{X}.\,\omega(\tilde{Y}) - \omega([\tilde{X}, \tilde{Y}]) = 0$$

$$L_{\tilde{Y}}\omega(\tilde{X}) = \tilde{Y}.\,\omega(\tilde{X}) - \omega([\tilde{Y}, X]) = 0$$

As a consequence we get, for example, that for a translation invariant connection of a Yang-Mills theory all the connection and curvature elements will be constant; therefore, all the subspaces m_k of the Lie algebra of the infinitesimal holomony group are obtained by

$$m_0 = \{[\tilde{\Gamma}_\mu X^\mu, \tilde{\Gamma}_\nu Y^\nu]; X, Y \text{ on } V\}$$

$$m_1 = m_0 + [\bar{\omega}(X), m_0]$$

$$m_2 = m_0 + [\bar{\omega}(X), m_0] + [\bar{\omega}(X), [\bar{\omega}(X), m_0]]$$

and so on, since

$$\tilde{\nabla}_X \tilde{R}_{\mu\nu} = [\tilde{\Gamma}_\rho X^\rho, \tilde{R}_{\mu\nu}]$$

$\bar{\omega}$: $\mathbf{T}_e{}^4 \to \mathbf{G}_e$ is the linear mapping, defined in Lemma 5, of the Lie algebra of the translation group into the Lie algebra of the structure group.

More general P-symmetries of a Yang-Mills bundle P induce some consequences on the spaces $m_k(u)$, defined in Section 4. Let $\tilde{X}$ be the generator of a $\tilde{f}$-symmetry, and $\tilde{V}$, $\tilde{W}$ arbitrary vector fields on P:

$$\begin{aligned}\tilde{X}\,.\,\Omega(\tilde{V}, \tilde{W}) &= L_{\tilde{X}}\Omega(\tilde{V}, \tilde{W}) + \Omega([\tilde{X}, \tilde{V}]),\ \ \tilde{W} + \Omega(\tilde{V}, [\tilde{X}, \tilde{W}]) \\ &= \Omega([\tilde{X}, \tilde{V}], \tilde{W}) + \Omega(\tilde{V}, [\tilde{X}, \tilde{W}])\end{aligned} \tag{5.3}$$

In order to relate $\tilde{X}\,.\,\Omega$ to the covariant derivative in the direction X we need the identities of Lemma 2:

$$\text{hor}\,\tilde{X}_u\,.\,\Omega = -\,\text{ver}\,\tilde{X}_u\,.\,\Omega + \tilde{X}_u\,.\,\Omega$$

$$\text{ver}\,\tilde{X}_u\,.\,\Omega = -\,[\omega_u(\tilde{X}), \Omega]$$

$$\text{hor}\,\tilde{X}_u\,.\,\Omega = \tilde{X}_u\Omega + [\omega_u(\tilde{X}), \Omega], \qquad \omega_u(\tilde{X})\ \text{const.}$$

or

$$\tilde{\nabla}_X \tilde{R}_{\mu\nu} = \tilde{X}\,.\,\tilde{R}_{\mu\nu} + [\tilde{\Gamma}_X, \tilde{R}_{\mu\nu}], \qquad X\,.\,\tilde{R}_{\mu\nu}\ \text{given by (5.3)}$$

Lemma 7. The direction derivative of the curvature in the direction X of a symmetry transformation is a linear combination of the curvature elements themselves, or

$$\tilde{X}\,.\,m_0(u) \subseteq m_0(u)$$

and the corresponding covariant derivative in this direction satisfies

$$\tilde{\nabla}_X \tilde{R}_{\mu\nu} = X\,.\,\tilde{R}_{\mu\nu} + [\tilde{\Gamma}_X, \tilde{R}_{\mu\nu}], \qquad \tilde{\Gamma}_X = X^\mu \tilde{\Gamma}_\mu$$

is constant.

Similar relations hold for higher order elements in the infinitesimal holonomy group: $\tilde{f} \in m_k(u)$ of the form (Kabayashi & Nomizu, 1963)

$$\tilde{f} = \tilde{X}_{\mu_1} \ldots \tilde{X}_{\mu_k} \Omega(\tilde{V}, \tilde{W})$$

$\tilde{X}_{\mu_i}$ are horizontal lifts of ∂_μ.

Let $\tilde{X}$ be a generator of the P symmetry, then $[\tilde{X}, \tilde{X}_{\mu_i}]$ is horizontal, i.e.

$$\tilde{X}\,.\,\tilde{X}_{\mu_i} = \tilde{X}_{\mu_i}\,.\,\tilde{X} + \tilde{Z}_{\mu_i}, \qquad \tilde{Z}_{\mu_i}\ \text{is horizontal}$$

$$\begin{aligned}\tilde{X}\,.\,(\tilde{X}_{\mu_1} \ldots \tilde{X}_{\mu_k}\Omega(\tilde{V}, \tilde{W})) &= \tilde{X}_{\mu_1}\,.\,(\tilde{X}\tilde{X}_{\mu_2} \ldots \tilde{X}_{\mu_k}\Omega(\tilde{V}, \tilde{W})) \\ &\quad + \tilde{Z}_{\mu_1}\tilde{X}_{\mu_2} \ldots \tilde{X}_{\mu_k}\Omega(\tilde{V}, \tilde{W})\end{aligned}$$

where the second term is of type (k); by permutation of the $\tilde{X}_{\mu_i}$ with $\tilde{X}$, we get

$$\tilde{X}\,.\,(\tilde{X}_{\mu_1} \ldots \tilde{X}_{\mu_k}\Omega(\tilde{V}, \tilde{W})) = \tilde{X}_{\mu_1} \ldots \tilde{X}_{\mu_k}(\tilde{X}\,.\,\Omega(\tilde{V}, \tilde{W})) + \tilde{g}, \qquad \tilde{g} \in m_k(u)$$

Therefore, together with Lemma 7 we have shown

Lemma 8. The subspaces $m_k(u)$ of the Lie algebra of the infinitesimal holonomy group are stable under these symmetry transformations generated by $\tilde{X}$

$$\tilde{X} . m_k(u) \subseteq m_k(u), \qquad \forall k = 0, 1, \ldots$$

or, in other words, if $\tilde{f}$ is of type

$$f_{\mu_1 \ldots \mu_k \rho\sigma} = \tilde{\nabla}_{\mu_1} \ldots \tilde{\nabla}_{\mu_k} \tilde{R}_{\rho\sigma}$$

then $X . f_{\mu_1 \ldots \mu_k \rho\sigma}$ can be expressed with the aid of these elements themselves.

In the following we look a little closer to the bundle symmetries which are induced by a group of isometries on Minkowski space-time, say I, this means that there exists a group of transformations, $\tilde{I}$, in P which transforms the fibre $\pi^{-1}(x)$ into the fibre $\pi^{-1}(f_V(x))$, $f_V \in I$. Let ψ be an arbitrary point in P with $\pi(\psi) = x_0$, and take those elements of I, which leave invariant the point x_0, $h(x_0) = x_0$; the set of these transformations build the stability group I_{x_0} at the point x_0. Now, in general, to $h \in I_{x_0}$ there corresponds an element in $\tilde{I}$, say h_P, such that

$$h_P\colon \psi(x_0) \to h_P(\psi(h(x_0))) = h_P(\psi(x_0))$$

with $h_P(\psi(x_0)) \in \pi^{-1}(x_0)$, since h_P is a bundle automorphism which does not alter the structure of the fibre. Therefore, we can find an element g in the structure group such that

$$h_P(\psi(x_0)) = \psi(x_0)g, \qquad g \in G$$

or, in other words, this defines a mapping $\bar{h}\colon I_{x_0} \to G$, $\bar{h}(h) = g$, which turns out a homomorphism since, if $h, h' \in I_{x_0}$ with the corresponding $h_P, h_{P'} \in \tilde{I}$

$$\begin{aligned}(h_P \circ h'_P)\,(\psi(x_0)) &= h_P(h'_P(\psi(x_0))) = h_P(\psi(x_0)g') = h_P(\psi(x_0)\bar{h}(h')) \\ &= (h_P(\psi(x_0)))\bar{h}(h') = (\psi(x_0)g)h(h') \\ &= (\psi(x_0)\bar{h}(h)\bar{h}(h') = \psi(x_0)(\bar{h}(h)\bar{h}(h'))\end{aligned}$$

on the other side, $h_P \circ h'_P = h''_P$

$$\begin{aligned}(h_P \circ h'_P)(\psi(x_0)) &= \psi(x_0)g'' = \psi(x_0)\bar{h}(h'') \\ &= \psi(x_0)(\bar{h}(h \circ h'))\end{aligned}$$

and therefore

$$\bar{h}(h \circ h') = \bar{h}(h) \cdot \bar{h}(h')$$

where the first product, $h \circ h'$ is the product in I, and the second product, $\bar{h}(h) \cdot \bar{h}(h')$ is the product in the structure group G. $\bar{h}$ depends in general on the chosen $\psi(x_0)$.

As an illustration we shall find the mapping $\bar{h}$ in the case of the Lorentz frame bundle P_4 over a general space-time V, where I is now the group of isometries of V. For a fixed x_0, an element $h \in I_{x_0}$ will generate a linear transformation h_* in the tangent space $T_x V$

$$(h_* X_{x_0})^\mu = \frac{\partial h^\mu}{\partial x^\nu}\bigg|_{x_0} X^\nu_{x_0}; \qquad X_{x_0} = X^\mu_{x_0} \partial_\mu \tag{5.4}$$

if the element h is represented locally by $x'^\mu = h^\mu(x^\nu)$. (5.4) is called the linear representation of the stability group at x_0, say $\Lambda^\mu_\nu(x_0) = \partial h^\mu / \partial x^\nu|_{x_0}$. Therefore, h transforms the Lorentz frames at the point x_0, $u_0 = (x_0, X_a(x_0))$

$$h_P(X_a(x_0)) = h_* X_a = (\Lambda^\mu_\nu(x_0) X^\nu_a(x_0))\partial_\mu$$

and we can find the corresponding $g \in G_4$ by

$$h_P(X_a(x_0)) = X_b(x_0) g^b{}_a$$

or

$$\Lambda^\mu{}_\nu(x_0) X_a{}^\nu(x_0) = X_b{}^\mu(x_0) g^b{}_a; \qquad \text{if } X_a{}^\mu Y^b{}_\mu = \delta^b{}_a$$

$$g^b{}_a = Y^b{}_\mu(x_0) \Lambda^\mu{}_\nu(x_0) X^\nu{}_a(x_0) \equiv \Lambda^b{}_a(x_0)$$

which defines our mapping $\bar{h}$: $I_{x_0} \to G_4$, since $\Lambda^b{}_a$ are the tetrad components of $\Lambda^\mu{}_\nu$. The explicit form for $\bar{h}$ shows indeed that $\bar{h}$ is a homomorphism.

For a second example, we look at a Yang-Mills bundle over Minkowski space-time; let I be especially a realisation of the rotation group in Minkowski space-time V such that the surfaces of transitivity are 2-spheres S^2. Take an $x_0 \in S^2$, defined by x^0 = const., r = const., and a $u_0 \in \pi^{-1}(x_0)$ with $u_0 = (x_0, \psi_a(x_0))$ where the ψ_a's transform at the same time covariantly under a representation of the Lorentz group, say U; ϕ_m span a basis of this representation space, $m = 1, 2, \ldots, M$, M = dimension of the representation. Then every element $\psi_a(x_0)$ of the Yang-Mills frame can be decomposed

$$\psi_a(x_0) = \psi^m_a(x_0)\phi_m, \qquad a = 1, \ldots, N$$

Every element $h \in I_{x_0}$ generates a transformation of the Yang-Mills frame

$$h \to U(h)\psi_a = (U(h)^m{}_{m'} \psi_a{}^{m'}(x_0))\phi_m, \qquad h(x_0) = x_0$$

which defines the mapping h_P. Therefore, we can find an element $g \in G$, such that

$$U(h)\psi_a(x_0) = \psi_b(x_0) g^b{}_a$$

by defining elements $\chi^a{}_m(x_0)$

$$\chi^a{}_m(x_0)\psi^m{}_b(x_0) = \delta^a{}_b$$

$$g^b{}_a = \chi^b{}_m(x_0) U(h)^m{}_{m'} \psi^{m'}{}_a(x_0) = \bar{h}(h)$$

Since the representation U is by definition a homomorphism, $U(h \circ h') = U(h)U(h')$, $\bar{h}(h)$ turns out to be also a homomorphism $I_{x_0} \to G$.

In Lemma 5 we defined a linear mapping $\bar{\omega}: \mathscr{H}(I) \to \mathbf{G}_e$ with respect to a local cross-section $(x, \psi_a(x))$; denote by $\bar{h}_*$ the mapping for the corresponding Lie algebras, i.e.

$$\bar{h}_*:\mathscr{H}(I_{x_0})\to \mathrm{G}_e \qquad \text{for fixed point } x_0\in V$$

If $X\in\mathscr{H}(I)$, X generates a transformation in V, f_t, and a corresponding transformation f_{tP} for the fibres with

$$\psi_t(x_0) = f_{tP}(\psi(x_0)) = \psi(x_0)\bar{h}(f_t)$$

then for the tangent vectors, which are completely vertical, we find

$$\omega_\psi(\dot{\psi}_t) = \bar{\omega}(X) = \bar{h}_*(X), \qquad X\in\mathscr{H}(I_{x_0})$$

This shows that the mapping $\bar{\omega}$ reduces, for an element of the Lie algebra of the stability group, to the above-defined mapping $\bar{h}_*$. Let the connection Γ be invariant under the transformation group $\tilde{I}$ induced by a group of isometries I. Fix $x_0\in V$ and $u_0 = (x_0, \psi_a(x_0))\in P$; take an $X\in\mathscr{H}(I)$ and an element $h\in I_{x_0}$; the 1-parameter group of transformations $h\circ f_t\circ h^{-1}, f_t$ generated by X, is generated by $adh(X)$ and transforms u_0 into a new frame

$$\begin{aligned}(h_P\circ\tilde{f}_{tP}\circ h_P{}^{-1})(\psi_a(x_0))&\\ &= (h_P\circ\tilde{f}_{tP})(\psi_a(x_0)\bar{h}(h^{-1}))\\ &= (h_P\circ\tilde{f}_{tP})(R_{\bar{h}(h-1)}\psi_a(x_0))\end{aligned}$$

Since Γ is assumed to be invariant under the transformation group $\tilde{I}$, we know from Lemma 1 that $\omega(\tilde{X})$ is constant, where $\tilde{X}$ generates $\tilde{f}_{tP}$. The problem is now to find a relation between $\omega_\psi(\tilde{X})$ and $\omega_\psi(\tilde{Y})$, where $\tilde{Y}$ generates $h_P\circ\tilde{f}_{tP}\circ h_P{}^{-1}$:

$$\tilde{Y}_\psi = h_{P*}(R_{\bar{h}(h^{-1})*}\tilde{X}_\psi)$$

and therefore

$$\begin{aligned}\omega_\psi(\tilde{Y}) &= \omega_\psi(h_{P*}(R_{\bar{h}(h^{-1})*}\tilde{X}_\psi))\\ &= \omega_{h_P^{-1}(\psi)}(R_{\bar{h}(h^{-1})*}\tilde{X}_\psi)\\ &= (ad\bar{h}^{-1}(h^{-1}))\omega_\psi(\tilde{X}_\psi) = ad\bar{h}(h)\omega_\psi(\tilde{X}_\psi)\end{aligned} \tag{5.5}$$

or, if we express ω with respect to the local cross-section, as in Lemma 4, this means

$$\tilde{\Gamma}_\mu Y^\mu = (ad\bar{h}(h))\tilde{\Gamma}_\alpha X^\alpha \tag{5.6}$$

X is the direction for the transformation f_t, Y is the direction rotated by the stability group element $h\in I_{x_0}$. Therefore, as an important consequence, we obtain the following result: the connection coefficients $\tilde{\Gamma}_\mu$ evaluated in the direction of space-time isometries have not to be 'isotropic' if the application $\bar{h}$ is non-trivial. This result is not in agreement with the result for spherically symmetric Yang–Mills connections obtained by Ikeda & Miyachi (1962) and later proved by Loose (1965): analytic spherically symmetric solutions of the

Yang-Mills equations (3.9) should admit only an Abelian holonomy group (which, in this case, is identical with the infinitesimal holonomy group).

The former proof in Loos (1965) was based on the following assumptions:

(1) the Yang-Mills connection is invariant under a realisation of the rotation group $SO(3)$ in Minkowski space-time; Γ is also assumed to be static;

(2) we look at a hypersurface x^0 = constant, and introduce on it spherical coordinates with r = constant defining a 2-sphere.

The transformation property (5.5) for the connection form under transformations of the isotropy group does in general not imply that $\Omega(\tilde{X}, \tilde{Y})$, $X \in T_{x_0}S^2$, $Y \in T^{\perp}_{x_0}S^2$, is independent of the direction of X as it was assumed to be the case in Ikeda & Miyachi (1962). If the Yang-Mills connection is 'isotropic' in the sense that $\bar{h}$ is trivial, then surely, we can conclude for the Yang-Mills curvature

$$\tilde{R}_{0\theta} = \tilde{R}_{0\phi} = \tilde{R}_{r\theta} = \tilde{R}_{r\phi} = 0$$

and then the Bianchi identities and the Yang-Mills equations imply immediately that the Lie algebra of the infinitesimal holonomy group, spanned by the elements of type (4.2), is commutative.

Since the Schwarzschild solution of Einstein's equations is indeed a solution of a Yang-Mills type equation with spherical symmetry—in this case determining the external geometry of a space-time—this example represents in some sense a solution of spherically symmetric Yang-Mills equations with non-Abelian infinitesimal holonomy group.

Particular solutions to Yang-Mills equations over Minkowski space-time can be found in Kerbrat, 1970; Ikeda & Miyachi, 1962; Loos, 1965, 1967; and Uzes, 1969.

References

Ikeda, M. and Miyachi, Y. (1962). *Progress of Theoretical Physics,* **27**, 474.

Kerbrat, H. Mme. (1970). *Annales de l'Institut Henri Poincaré,* **13**, 295, and references therein.

Kobayashi, S. and Nomizu, K. (1963). *Foundations of Differential Geometry,* Vol. I. Interscience, New York.

Lichnerowicz, A. (1962). *Théorie Globale des Connexions et des Groupes d'Holonomie, Cremonese.*

Loos, H. G. (1965). *Nuclear Physics,* **72**, 677.

Loos, H. G. (1967). *Journal of Mathematical Physics,* **8**, 1870, 2114.

O'Raifeartaigh, L. (1970). Unitary Representations of Lie Groups in Quantum Mechanics; in Group Representations in Mathematics and Physics, Battelle 1969 Rencontres, *Springer Lecture Notes in Physics,* Vol. 6.

Trautman, A. (1970). *Reports on Mathematical Physics,* **1**, 27.

Utiyama, R. (1956). *Physical Review,* **101**, 1597.

Uzes, C. A. (1969). *Journal of Mathematical Physics,* **10**, 1885.

Uzes, C. A. (1971). *Journal of Mathematical Physics,* **12**, 716.

Yang, C. N. and Mills, R. (1954). *Physical Review,* **96**, 191.